Main Theorem. X smooth projective curve $/\, k = \bar{k}$ char 0 w/ genus $= g \geq 2$.

$M_X^{ss}(r, d)$, $M_X^{ss}(r, L)$ are projective varieties over k

with dimensions $r^2(g-1)+1$, $(r^2-1)(g-1)$.

Recall Six steps toward projective moduli (Section 0.7.2)

$\mathcal{M}$ = moduli stack for our moduli problem

Step 1 Algebraicity : Express $\mathcal{M}$ as a substack of a larger moduli stack $\mathcal{X}$.

$\mathcal{E} \in \mathcal{X}$ is semistable $\iff \mathcal{E} \in \mathcal{M}$

Show $\mathcal{X}$ is an alg. stack locally of finite type.

Step 2 Openness of semistability : Show $\mathcal{M} \overset{\text{open}}{\subset} \mathcal{X}$.

$(\iff \forall \mathcal{E} \in \mathcal{X}$ over scheme $S : \{s \in S \mid \mathcal{E}_s$ is semistable$\}$ is open)

Step 3 Boundedness of semistability : Show $\mathcal{M}$ is of finite type.

Step 4 Semistable reduction : Show that $\mathcal{M}$ satisfy some

valuative criteria for some separation notion.

Step 5 Existence of moduli space : Show that there exists a

fine / coarse / good moduli space $\mathcal{M} \to M$ where M is

a proper algebraic space.

Step 6 Projectivity : Show that a tautological line bundle on $\mathcal{M}$

descends to an ample line bundle on M.

Today : (Steps 1 – 3)

- Review of rank, degree of VBs over a curve, algebraicity of moduli stacks
- Openness, boundedness, irreducibility of semistability

Degree & rank of coherent sheaves over a curve

Def. X a projective curve $/k$, F coherent sheaf on X.

$\operatorname{rank} F := \dim_{k(\eta)} F_\eta$ (η = generic point of X)

$\deg F := \chi(X, F) - (\operatorname{rank} F) \cdot \chi(X, \mathcal{O}_X)$

Some properties:

- If E is a VB, $\deg E = \deg(\det E)$
- Rank and degree are additive on exact sequence.
- $\operatorname{rank}(F \oplus F') = \operatorname{rank} F + \operatorname{rank} F'$, $\operatorname{rank}(F \otimes F') = \operatorname{rank} F \cdot \operatorname{rank} F'$
- $\deg(F \oplus F') = \deg F + \deg F'$

  If E, E' are VBs, $\deg(E \otimes E') = \operatorname{rank} E' \cdot \deg E + \operatorname{rank} E \cdot \deg E'$

  $\left(\because \det(E \otimes E') \cong (\det E)^{\operatorname{rank} E'} \otimes (\det E')^{\operatorname{rank} E}\right)$
- Assuming X is smooth and taking $\mathcal{O}_X(1)$ as an invertible sheaf with $\deg 1$, $\forall$ VB E: $P_E(n) = \operatorname{rank} E \cdot n + \deg E + \operatorname{rk} E \cdot \chi(X, \mathcal{O}_X)$.

Algebraicity of moduli stacks of VBs

X smooth projective curve $/k = \bar{k}$ char 0. (genus $= g$)

The following are all stacks over Sch_k.

$\underline{\mathrm{Coh}}_X$: Object $/S$ = coherent sheaves on $X \times_k S$ flat over S

Morphism $(F', S') \to (F, S)$ over $S' \to S$

$=$ map $F \to (\mathrm{id}_X \times f)_* F'$ s.t. $(\mathrm{id}_X \times f)^* F \xrightarrow{\cong} F'$.

M_X: stack of vector bundles

$M_X(r, d)$: stack of vector bundles with rank r and degree d

(i.e. $E \in \mathrm{Mod}(\mathcal{O}_{X \times_k S})$ object over S.

$\forall$ geometric point $\operatorname{Spec} K \to S$:

$E|_{X \times \operatorname{Spec} K}$ has rank r, degree d)

Prop. $\mathcal{M}_X(r,d)$ is an alg. stack w/ affine diagonal and loc. of finite type/k.

pf $\underline{\mathrm{Coh}}_X$ is $\simeq$. ([ODLY], [ODLZ])

Suffices to show $\mathcal{M}_X(r,d) \overset{\text{open}}{\subseteq} \underline{\mathrm{Coh}}_X$.

- $\mathcal{M}_X \overset{\text{open}}{\subseteq} \underline{\mathrm{Coh}}_X$: Given F coherent sheaf on $X \times S$ flat over S, locus $Z \subseteq X \times S$ where F is not locally free is closed.
  $\Rightarrow \{ s \in S \mid F|_{X \times \{s\}} \text{ locally free} \} = S \setminus \mathrm{pr}_2(Z) \overset{\text{open}}{\subseteq} S$.
- $\mathcal{M}_X(r,d) \overset{\text{open}}{\subseteq} \mathcal{M}_X$ (also closed) : The Hilbert polynomial is locally constant in flat families.
  $\forall E$ VB on X : $P_E(n) = \mathrm{rk}(E)\,n + \deg(E) + \mathrm{rk}(E)(1-g)$. ∎

Let L a line bundle on X. ($\deg L = d$)

$\mathcal{M}_X(r,L)$: stack of vector bundles s.t. $\det = L$. (and rank $= r$)

$\mathrm{Object}/S = (E, \varphi)$ s.t. $\begin{cases} E \text{ VB on } X \times S \text{ w/ rank } = r \text{ (also flat/} S) \\ \varphi : \det E \xrightarrow{\cong} L|_{X \times S} \end{cases}$

Facts. (1) $\exists \det : \mathcal{M}_X(r,d) \longrightarrow \mathcal{P}ic^d_X := \mathcal{M}_X(1,d)$
$(E \mapsto \det E)$

$\forall L \in \mathcal{P}ic^d_X$:
$$\begin{array}{ccc} \mathcal{M}_X(r,L) & \longrightarrow & \mathcal{M}_X(r,d) \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}\, k & \xrightarrow{[L]} & \mathcal{P}ic^d_X \end{array} \qquad (!)$$

(2) $\exists$ "universal VB" $\mathcal{E}_{\mathrm{univ}}$ on $X \times \mathcal{M}_X(r,d)$
s.t. $\forall E$ object$/S$ of $\mathcal{M}_X(r,d)$: $E \cong (\mathrm{id}_X \times [E])^* \mathcal{E}_{\mathrm{univ}}$.

In particular, we have "Poincare VB" $\mathcal{P}$ on $X \times \mathcal{P}ic^d_X$.

$\Rightarrow (\mathrm{id}_X \times \det)^* \mathcal{P} \cong \det(\mathcal{E}_{\mathrm{univ}})$.

Prop. $\mathcal{M}_X(r,L)$ is an alg. stack w/ affine diagonal and loc. of finite type/k.

pf Immediate from the cartesian diagram (!).

Tangent spaces of moduli stacks

The following are consequences of deformation theory.

Prop. Assume $g \geq 2$.

(1) $\mathcal{M}_X(r, d)$, $\mathcal{M}_X(r, L)$ are smooth over k.

(2) Let E VB on X with rank r, degree d. $\varphi : L \xrightarrow{\cong} \det E$.

$$T_{\mathcal{M}_X(r,d),\,[E]} \cong \mathrm{Ext}^1(E, E)$$

$$T_{\mathcal{M}_X(r,L),\,[(E,\varphi)]} \cong \ker\left(\mathrm{Ext}^1(E, E) = H^1(X, E^\vee \otimes E) \xrightarrow{\mathrm{tr}} H^1(X, \mathcal{O}_X)\right)$$

(cf. Proposition 3.7.5)

Remark. (Dimension calculation)

Suppose E is simple. (i.e. $\dim \mathrm{Hom}_X(E, E) = 1$. e.g. stable VBs)

$$\begin{aligned}
\dim T_{\mathcal{M}_X(r,d),\,[E]} &= \dim H^1(X, E^\vee \otimes E) \\
&= \dim H^0(X, E^\vee \otimes E) - \chi(X, E^\vee \otimes E) \\
&= \dim \mathrm{Hom}_X(E, E) - \deg(E^\vee \otimes E) - \mathrm{rk}(E^\vee \otimes E)(1 - g) \\
&= r^2(g - 1) + 1.
\end{aligned}$$

Since we're working on char 0, $\mathrm{tr} : E^\vee \otimes E \to \mathcal{O}_X$ splits by the section $\frac{1}{r}\,\mathrm{id}_E : \mathcal{O}_X \to \mathrm{Hom}_X(E, E) \xrightarrow{\cong} E^\vee \otimes E$.

$$\begin{aligned}
\dim T_{\mathcal{M}_X(r,L),\,[(E,\varphi)]} &= \dim H^1(X, E^\vee \otimes E) - \dim H^1(X, \mathcal{O}_X) \\
&= (r^2 - 1)(g - 1).
\end{aligned}$$

Stability notions and properties

Def. E nonzero VB on X. **Slope** of E $\mu(E) := \frac{\deg(E)}{rk(E)}$.

(※ Keep in mind that any subsheaf of a VB is a VB!)

Def. A VB E on X is

- **semistable** if $\forall$ nonzero $E' \subset E$: $\mu(E') \leq \mu(E)$
- **stable** if semistable & the only $E' \subset E$ s.t. $\mu(E') = \mu(E)$ is E
- **polystable** if it is isomorphic to $\bigoplus_i E_i$ where E_i stable w/ $\mu(E_i) = \mu(E)$

Lemma. (Properties of stability notions)

(1) L LB, E VB. E (semi) stable $\Leftrightarrow E \otimes L$ (semi) stable

(2) E, F semistable VBs w/ $\mu(E) > \mu(F) \Rightarrow \mathrm{Hom}_X(E, F) = 0$

(3) $0 \to E_1 \to E \to E_2 \to 0$ SES of VBs.

E_1, E_2 semistable and $\mu(E_1) = \mu(E_2) \Rightarrow E$ semistable

(4) E a stable VB $\Rightarrow E$ is simple (i.e. $\mathrm{End}_X(E) \cong k$)

(5) E a semistable VB s.t. $\mu(E) > 2g - 1$.

Then E is globally generated and $H^1(X, E) = 0$.

pf (1) $\deg(E \otimes L) = \deg(E) + rk(E) \cdot \deg(L)$.

$\Rightarrow \mu(E \otimes L) = \mu(E) + \deg(L)$.

So $\forall E' \subset E$: $\mu(E) - \mu(E') = \mu(E \otimes L) - \mu(E' \otimes L)$.

(2) Suppose $E \to F$ nonzero map. Let kernel $= K$, image $= I$.

We have $0 \to K \to E \to I \to 0$ SES. Assume $K \neq 0$.

$$\Rightarrow \mu(I)\,rk(I) = \mu(E)\,rk(E) - \mu(K)\,rk(K)$$
$$\geq \mu(E)\,rk(E) - \mu(E)\,rk(K) \quad (\because \mu(K) \leq \mu(E))$$
$$= \mu(E)\,rk(I).$$

Since $rk(I) > 0$, $\mu(I) \geq \mu(E)$. But $\mu(E) > \mu(F)$ and $\mu(F) \geq \mu(I)$, contradiction.

(3) Let $\mu := \mu(E_1) = \mu(E_2)$. Then $\mu(E) = \mu$.

Take nonzero $E' \subset E$. Let $E_1' = E_1 \cap E'$, $E_2' = Im(E' \hookrightarrow E \to E_2)$.

$\Rightarrow \exists\ 0 \to E_1' \to E' \to E_2' \to 0$ SES.

$$\begin{cases} E_1' = 0 \text{ or } \mu(E_1') \leq \mu \\ E_2' = 0 \text{ or } \mu(E_2') \leq \mu \end{cases} \Rightarrow \mu(E') \leq \mu.$$

(4) Take any nonzero map $f: E \to E$. Following proof of (2), f is iso.

Pick any $x \in X$. If λ is an eigenvalue of f_x,

$f - \lambda id_X \in End_X(E)$ is not iso, so it's zero. $\Rightarrow f = \lambda id_X$.

(5) $H^1(X, E) = H^0(X, E^\vee \otimes \omega_X) = Hom_X(E, \omega_X) = 0$.

$$(\because \mu(E) > \mu(\omega_X) = 2g - 2)$$

$\forall x \in X(k)$ closed point: By Nakayama's lemma, it suffices to show

$$\Gamma(X, E) \to E_x \to E_x / m_{X,x} E_x \text{ is surj.}$$

Let $\mathcal{O}_x$ = skyscraper sheaf at x w/ value $\kappa(x)$.

$\Rightarrow \exists$ SES $0 \to E \otimes \mathcal{O}_x(-x) \to E \to \mathcal{O}_x^{\oplus rk(E)} \to 0$.

Above map is map of global sections of $E \to \mathcal{O}_x^{\oplus rk(E)}$,

so it suffices to show $H^1(X, E \otimes \mathcal{O}_x(-x)) = 0$.

$H^1(X, E \otimes \mathcal{O}_x(-x)) = Hom_X(E \otimes \mathcal{O}_x(-x), \omega_X) = 0$.

$$(\because \mu(E \otimes \mathcal{O}_x(-x)) = \mu(E) - 1 > \mu(\omega_X) = 2g - 2)$$

Filtrations of VBs

Def/Prop. (Jordan – Hölder filtration)

E semistable VB on X.

$\Rightarrow$ $\exists$ filtration $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell = E$

s.t. each subquotient $G_i = E_i / E_{i-1}$ is stable w/ $\mu(G_i) = \mu(E)$.

For any two such filtrations, the multiset of subquotients are same up to isomorphism. Define the <u>associated graded bundle</u> $gr_E := \bigoplus_{i=1}^{\ell} G_i$.

Def. E, E' semistable VBs w/ same rank & degree are called <u>S-equivalent</u> if $gr_E \cong gr_{E'}$.

Def/Prop. (Harder – Narasimhan filtration)

E VB on X.

$\Rightarrow$ $\exists!$ filtration $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell = E$

s.t. each subquotient $G_i = E_i / E_{i-1}$ is semistable

and $\mu(G_1) > \mu(G_2) > \cdots > \mu(G_\ell)$.

Ex. Let $X = \mathbb{P}^1$. Any VB E on X splits as a direct sum of LBs.

$E \cong \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^1}(d_i)^{\oplus m_i}$ w/ $d_1 > d_2 > \cdots > d_\ell$

$\Rightarrow G_i = \mathcal{O}_{\mathbb{P}^1}(d_i)^{\oplus m_i}$. $(\mu(G_i) = d_i)$

In particular, $\begin{cases} E \text{ semistable} \iff \ell = 1 \\ E \text{ stable} \iff E \text{ is a LB} \end{cases}$

Openness, boundedness, irreducibility of semistability

Prop. (Openness) $\mathcal{M}_X^{ss}(r,d) \overset{\text{open}}{\subseteq} \mathcal{M}_X(r,d)$

pf: Equivalent to: If S is a k-scheme and $\mathcal{E}$ a VB w/ rank r and degree d on $X \times S$.

Then $\{p \in S \mid \mathcal{E}|_{X\times\{p\}} \text{ is semistable}\} \overset{\text{open}}{\subseteq} S$.

- Reduction to the case S finite type $/k$ and connected:

  By A.6.2 & A.6.8, $S = \varprojlim S_\lambda$ s.t. S_λ finite type $/k$ w/ affine transition maps

  & $\exists \lambda$ s.t. $\mathcal{E} \cong (\mathrm{id}_X \times (S \to S_\lambda))^* \mathcal{E}_\lambda$ for some VB $\mathcal{E}_\lambda$.

  Reduction to connected case is easy.

- Express un-semistable locus as union of images of Quot schemes:

  $\forall p \in S$: $\mathcal{E}|_{X\times\{p\}}$ is not semistable

  $\iff \exists E' \subseteq \mathcal{E}|_{X\times\{p\}}$ s.t. $\mu(E') > \frac{d}{r}$

  $\iff \mathcal{E}|_{X\times\{p\}}$ has a quotient with Hilbert polynomial $P_{\mathcal{E}}(n) - P_{E'}(n)$ for some $E' \subseteq \mathcal{E}|_{X\times\{p\}}$ s.t. $\mu(E') > \frac{d}{r}$.

  Hence the set of such $p \in S$ is $\bigcup_{Q(n)} \mathrm{Im}\left(\mathrm{Quot}^{Q(n)}_{\mathcal{E}/X\times S/S} \to X \times S \xrightarrow{p_2} S\right)$

  where $Q(n)$ runs through

  $\{P_{\mathcal{E}}(n) - P_{E'}(n) \mid \exists p \in S, \exists E' \subseteq \mathcal{E}|_{X\times\{p\}} \text{ s.t. } \mu(E') > \frac{d}{r}\}$.

• union is finite :

$P_{E'}(n)$ determined by $\text{rank}(E')$ & $\deg(E')$.

$\text{rank}(E')$ is obviously bounded, and $\deg(E')$ is bounded below since $\mu(E') > \frac{d}{r}$.

So suffices to show $\deg(E')$ is bounded above.

Choose a finite morphism $X \to \mathbb{P}^1$. $\Rightarrow h : X \times S \to \mathbb{P}^1 \times S$.

If $E' \subset \mathcal{E}_p$, then $h_* E' \subset h_* \mathcal{E}_p$. Note $\chi(\mathbb{P}^1, h_* E') = \chi(X, E')$.

Since $\begin{cases} \chi(\mathbb{P}^1, h_* \mathcal{E}_p) \text{ attains finitely many values} \\ \quad (\because h^0(\mathbb{P}^1, h_* \mathcal{E}_p),\ h^1(\mathbb{P}^1, h_* \mathcal{E}_p) \text{ attain finitely many values} \\ \quad\quad \text{by semicontinuity} \Rightarrow \text{finitely many splitting types of } h_* \mathcal{E}_p) \\ \text{for each VB on } \mathbb{P}^1, \text{ the Euler characteristics of} \\ \quad \text{all its subsheaves are bounded above,} \end{cases}$

$\chi(X, E')$ is bounded above. So $\deg(E')$ bounded above. ▨

Def. A collection of isomorphism classes of VBs on X is **bounded** if $\exists$ scheme of finite type S and VB F on $X \times S$ (flat over S) s.t. every isomorphism class $= [F_s]$ for some k-point $s \in S$.

Ex. Fix $x \in X$, $r \geq 2$, $d \in \mathbb{Z}$.

$\{ F_n = \mathcal{O}_X(-nx) \oplus \mathcal{O}_X((n+d)x) \oplus \mathcal{O}_X^{\oplus(r-2)} \}$ is an unbounded family which consists of fixed rank r, degree d VBs. ($\because h^1(X, F_n)$ unbounded)

Ex. Fix F VB on X, $P(n) \in \mathbb{Z}[n]$.

{VBs that are quotients of F w/ Hilbert polynomial $P(n)$} is bounded.

($\because$ restrict the universal quotient sheaf on $X \times \text{Quot}^{P(n)}_{F/X/\text{Spec } k}$ to $X \times$ (open locus of $\text{Quot}^{P(n)}_{F/X/\text{Spec } k}$ where pullback is locally free) to get a family of VBs realizing each class)

Prop. (Boundedness) The collection of isomorphism classes of semistable VBs of fixed rank r and degree d on X is bounded.

pf Let $x \in X$ a k-point & m an integer s.t. $\frac{d}{r} + m > 2g-1$.

For any such VB E, $E \otimes \mathcal{O}_X(mx)$ is semistable w/ slope $> 2g-1$.

$$\Rightarrow \begin{cases} E \otimes \mathcal{O}_X(mx) \text{ globally generated} \\ h^1(X, E \otimes \mathcal{O}_X(mx)) = 0 \Rightarrow h^0(X, E \otimes \mathcal{O}_X(mx)) =: N \text{ fixed.} \end{cases}$$

Let V a k-vector space w/ dimension N.

Then $\exists$ surjection $V \otimes_k \mathcal{O}_X \twoheadrightarrow E \otimes \mathcal{O}_X(mx)$, so $V \otimes_k \mathcal{O}_X(-mx) \twoheadrightarrow E$.

$\therefore$ Every E is a quotient of $V \times_k \mathcal{O}_X(-mx)$ w/ Hilbert polynomial $rn + (d+1-g)$. ∎

Prop. (Irreducibility) $\mathcal{M}_X^{ss}(r,d)$, $\mathcal{M}_X^{ss}(r,L)$ are irreducible.

pf

- **Reduction to the case $d > r(2g-1)$:**

  Fix $x \in X(k)$ closed point.

  E is semistable $\Leftrightarrow E \otimes \mathcal{O}_X(x)$ is semistable, so we have an isomorphism $\mathcal{M}_X^{ss}(r,d) \xrightarrow{\cong} \mathcal{M}_X^{ss}(r, d+r)$.

  So we can assume $d > r(2g-1)$, and all semistable bundles with rank r and degree d are globally generated.

- **Construct a map from an irreducible variety V^{ss} to $\mathcal{M}_X^{ss}(r,d)$:**

  Idea: take V^{ss} as a space that parameterize semistable E s.t. $\exists$ LB L & SES $0 \to \mathcal{O}_X^{r-1} \to E \to L \to 0$

For a fixed LB L, the extensions $0 \to \mathcal{O}_X^{r-1} \to E \to L \to 0$ are parameterized by $\mathrm{Ext}^1(L, \mathcal{O}_X^{r-1}) = H^1(X, L^\vee)$.

Let $\mathcal{L}$ the Poincare LB on $X \times \mathrm{Pic}_X^d$ and $\pi : X \times \mathrm{Pic}_X^d \to \mathrm{Pic}_X^d$ the projection.

By Grauert's theorem (A.7.4), $V = R^1\pi_* \mathcal{L}^\vee$ is a VB on Pic_d^X and the comparison map $R^1\pi_* \mathcal{L}^\vee \otimes k \to H^1(X, L^\vee)$ is iso.

$\Rightarrow$ the fiber of V over $L \in \mathrm{Pic}_X^d$ is $H^1(X, L^\vee)$

$\Rightarrow$ $\exists$ universal extension $\mathcal{E}_{univ}$ on $X \times V$ ("V" = $\underline{\mathrm{Spec}}_{\mathcal{O}_{\mathrm{Pic}_X^d}}(\mathrm{Sym}(V))$)

i.e. $\forall w \in V \mapsto L \in \mathrm{Pic}_X^d$: $\mathcal{E}_{univ}|_{X \times \{w\}} \cong E$.

(corr. to $0 \to \mathcal{O}_X^{r-1} \to E \to L \to 0$)

The locus $V^{ss} \subset V$ where $\mathcal{E}_{univ}$ is semistable is open.

So we obtain a map $V^{ss} \to \mathcal{M}_X^{ss}(r, d)$.

- $V^{ss} \to \mathcal{M}_X^{ss}(r, d)$:

Suffices to show every semistable VB E on X is an extension of LB by $\mathcal{O}_X^{r-1}$. (only shows surj on closed points, but sufficient by Chevalley's theorem)

I claim $\exists T \subset H^0(X, E)$ subspace of dim $r-1$

s.t. $ev : T \otimes \mathcal{O}_X \to E$ has rank $r-1$ on all points of X. (#)

$\forall x \in X$: kernel of $H^0(X, E) \to E_x$ has codimension r.

($\because$ E globally generated)

$\Rightarrow$ The locus of subspaces in $\mathrm{Gr}(r-1, H^0(X, E))$ which doesn't intersect the kernel nontrivially has codim 2.

$\Rightarrow$ The locus of T s.t. (#) is not true has codim 1, so we can take T s.t. (#) is true.

Then we have a SES $0 \to \Gamma \otimes \mathcal{O}_X \xrightarrow{ev} E \to \det(E) \to 0$.

$(\alpha \mapsto g_1 \wedge \cdots \wedge g_{r-1} \wedge \alpha)$

Proof for $\mathcal{M}_X^{ss}(r, L)$ is similar using an open substack of $\mathrm{Ext}^1_X(L, \mathcal{O}_X)$ surjecting onto the stack. ∎

<u>Remark.</u> If $g \geq 2$, $\forall r > 0$, $\forall L \in \mathrm{Pic}^d_X$: $\exists E$ VB w/ rank r and determinant L that is stable.

$\Rightarrow \mathcal{M}_X^s(r, d) \overset{\text{open}}{\subseteq} \mathcal{M}_X^{ss}(r, d)$, $\mathcal{M}_X^s(r, L) \overset{\text{open}}{\subseteq} \mathcal{M}_X^{ss}(r, L)$ are nonempty and dense.